One-dimensional Port-and-Sweep Solitaire Armies

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1 Introduction

Peg solitaire is a venerable puzzle system with a substantial mathematical theory and literature. The enthusiast who scratches the surface of the game will uncover a rich algebra of mod 3 invariants [1] [3] and an unexpected connection to Fibonacci numbers and the golden ratio in the problem known as "Conway's Soldiers" or the "Solitaire Army"[1] [2].

Port-and-Sweep Solitaire (PaSS) is a modern variant of peg solitaire, introduced in 2010 [6], which is also played with counters on a square grid. In this paper we will address the PaSS equivalent of the solitaire army problem in one dimension. The OPS solitaire army becomes rather dull when restricted to one dimension [5]. In PaSS, the situation seems considerably more difficult and interesting.

We will state the problem precisely after a quick introduction to the rules of PaSS. For those who are already familiar with the solitaire army problem, however, we can offer the following summary of our results:

1. A one-dimensional PaSS army can advance as much as seven steps. By way of comparison, a one-dimensional OPS army can advance no more than a single step [5].
2. A one-dimensional PaSS army can not advance 9 steps or more.
3. Under some mild additional restrictions on movement, we can show that an advance of 8 is impossible, but our proof does depend on those additional restrictions. We believe that an advance of 8 is extremely unlikely, but our results so far do not totally rule it out.

2 PaSS Rules

PaSS is best understood by comparison with ordinary peg solitaire. In OPS, each cell of the board may hold zero or one pegs. Moves are carried out physically by jumping one peg over an adjacent one into an empty cell, and removing the peg that was jumped. Equivalently, a move consists of adding | -1 | -1 | +1 | to the "peg count" of three consecutive squares on the grid – subject to the rule that a square can't contain more than one peg, or a negative number of pegs. This fundamental move can be applied in any direction on the board: up, down, left or right.

Similarly, PaSS has one rule constraining legal positions on the board:

- Each cell in PaSS may contain zero, one, or two counters.

PaSS has two types of move, which may be played in any direction:

- (Sweep move) Add | -1 | -1 | -1 | +2 | to four consecutive squares on the board

- (Port move) Add | -2 | 0 | +1 | to three consecutive squares on the board.

Each move has one or more *source* cells (colored) from which counters are deducted, and a single *target* cell where the count increases. The net change for either kind of move is a decrease of one counter on the board.

An interactive PaSS tutorial (including puzzles to solve) is available online [7], and you may wish to familiarize yourself with the rules by playing a bit before reading on.

3 PaSS Armies

In this paper, we limit our attention to boards consisting of a single row of cells forming a one-dimensional strip, which may be assumed to continue infinitely in both directions. We mark a line between two cells on the board and begin with some arrangement of finitely many counters to the left of the line (satisfying the usual requirement of no more than two counters per cell). Any such arrangement will be called an *army*. Our fundamental question is: *Beginning with any army of our choice, and proceeding by legal PaSS moves, how far to the right of the line can we place a counter?*

We can illustrate this by showing an army and sequence of moves which places a counter seven cells to the right of the line:

[figure: advance of 7]

The army, although finite, can be arbitrarily large, so it may seem surprising that there is any bound at all on the distance we can advance our counters. However, we believe that the advance of seven, shown above, is best possible. We present our results in the form of three propositions; the proofs will follow in the subsequent sections.

Theorem 1. *It is possible to advance a counter any number of cells to the right of the line, up to (and including) seven.*

We will prove this simply by presenting armies which achieve the necessary advance in each case. The most interesting is the advance of seven, which we have already seen. Upper bounds for the PaSS army problem are provided in the following theorems.

Theorem 2. *It is impossible to advance a counter nine or more cells to the right of the line.*

Examining the solution for the advance of seven, one notes that all the moves used are *rightward* moves – that is, the target cells lies to the right of the source cells. It seems intuitive that leftward moves would be counterproductive, and the optimum advance should use rightward moves only. Thus far, we are not able to prove this "intuitive" assertion. However, we do have the following.

Theorem 3. *It is impossible to advance a counter eight or more cells to the right of the line using exclusively rightward moves.*

Equivalently, if there were any advance of eight, it would have to use some leftward moves. The proof of Theorem 2 is relatively straightforward and mostly depends on ideas adapted from the OPS army problem. Theorem 3 is not so tidy and our proof depends, past a certain point, on a computer enumeration of possibilities.

4 Successful Advances

As stated above, we will prove that advances up to a distance of seven are all possible by displaying armies that attain those distances. Due to the relatively small number of moves (up to 14), these positions can be verified relatively easily. While there are infinitely many armies that can advance these given distances, we show those that do so with the minimum number of initial counters. Table 1 displays armies with the minimum number of counters to advance any distance up to seven spaces.

Distance	Army
1	0 0 0 0 0 0 2 0 | 0
2	0 0 0 0 0 0 0 2 | 0 0
3	0 0 0 0 0 1 1 1 | 0 0 0
4	0 0 0 0 2 0 2 1 | 0 0 0 0
5	0 0 0 0 0 2 2 2 | 0 0 0 0 0
6	0 0 0 2 2 2 2 2 | 0 0 0 0 0 0
7	1 2 2 2 2 2 2 2 | 0 0 0 0 0 0 0

Table 1: Minimum-counter armies that can advance up to seven spaces

The minimality of the above positions is not a critical assertion, but can be verified through a backtracking approach where one counter is placed on the desired space and moves are used in reverse to achieve a valid army in the least number of steps.

5 Values of Positions

In order to tackle the one-dimensional PaSS army problem, we use Conway's idea of a weight function [4], which assigns a value to any position of the board. To that end, define α to be the real root of the polynomial $-1 - x - x^2 + 2x^3$, which is approximately 1.23375. Since α is an algebraic number, computer algebra systems such as Mathematica can compute effectively with α in exact terms, and we take advantage of this in the subsequent proofs. Note that the coefficients of the defining polynomial represent the change in counters induced by a rightward sweep move.

Let S denote an arbitrary position of the board. We give the spaces of the board indices similar to the x-axis of the Cartesian plane, where all squares to the right of the line are given positive indices. Let $S(i)$ denote the number of counters on space with index i in S.

Define the *α-value* of a position S, denoted $\alpha(S)$, by the formula:

$$\alpha(S) = \sum S(i) \cdot \alpha^i$$

We're using α to denote both a number and a function of board position, but context should be sufficient to distinguish between the two uses. This α-value has a monotonicity property which is helpful in proving the impossibility of certain advances.

Proposition 1. *If position S' is obtained by a legal move from position S, then $\alpha(S') \leq \alpha(S)$.*

Proof. If the target cell of the move has index i, then the change in α-value $\alpha(S') - \alpha(S)$ is one of the following values:

1. $\alpha^i(2 - \alpha^{-1} - \alpha^{-2} - \alpha^{-3})$
2. $\alpha^i(1 - 2\alpha^{-2})$
3. $\alpha^i(2 - \alpha - \alpha^2 - \alpha^3)$
4. $\alpha^i(1 - 2\alpha^2)$

The first of the four quantities (corresponding to a rightward sweep) is zero by the definition of α; rightward sweeps do not change the value of a position. The remaining three quantities, corresponding to rightward port, leftward sweep, and leftward port respectively, are easily seen to be negative, whether by hand or by computer algebra (remembering that the α^i factor will always be positive since α itself is). Hence, $\alpha(S') \leq \alpha(S)$. □

Proposition 2. *If A is a valid starting army, then*

$$\alpha(A) \leq \frac{2\alpha}{\alpha - 1}.$$

Proof. Since A is an army, $A(i) = 0$ for every index $i > 0$, and $A(i) \leq 2$ for every index i by the rules of the game. Thus,

$$\alpha(A) = \sum_{i=0}^{\infty} A(i)\alpha^{-i} \leq \sum_{i=0}^{\infty} 2\alpha^{-i} = \frac{2\alpha}{\alpha - 1},$$

using the standard formula for the sum of a convergent geometric series. □

Let $\mathcal{M}$ denote this value, $2\alpha/(\alpha - 1)$, which serves as an upper bound for the α-value of any army. Numerically, $\mathcal{M}$ is approximately 10.55.

6 Impossible Advances

Propositions 1 and 2 together set an upper bound, $\mathcal{M}$, for the α-value of any position achievable by any starting army. We can use this value $\mathcal{M}$ to disprove some unrealistic positions described in the following proposition:

Proposition 3. *Let S be any position obtained by an initial army A, $S(i) = 0$ for $i \geq 12$ and $S(i) \leq 1$ for $i \geq 8$.*

Proof. If $S(i) > 0$ for some $i \geq 12$, then then $\alpha(S) \geq \alpha^{12}(\approx 12.44) > \mathcal{M}$. Similarly, if $S(i) > 1$ for $i \geq 8$, $\alpha(S) \geq 2\alpha^{8}(\approx 10.73) > \mathcal{M}$. Hence, no position S can have positive counters in cell 12 or above, or a double counter in cell 8 or above. □

Proof of Theorem 2. Proposition 3 immediately tells us that no initial army can advance a counter 12 or more spaces beyond the line.

Furthermore, if $S(11) = 1$, then a rightward port from space nine must have been made at some point. That indicates an earlier position S' achievable by a starting army with $S'(9) = 2$, contradicting Proposition 3. Using the same reasoning, we can easily show that for an initial army A with $S(10) = 1$, indicating a position S' achievable by a starting army with $S'(8) = 2$, which is also in contradiction of Proposition 3. Hence, no initial army can advance counters to cell ten or beyond.

While showing that an army cannot advance a distance of ten or beyond is mostly straightforward by the value $\mathcal{M}$, showing that an army cannot advance a counter up to cell nine is a bit trickier. We can immediately ignore the case where $S(9) = 2$ by Proposition 3, and begin by looking at the final position with $S(9) = 1$. We will show that the following set of moves must occur to achieve the final position:

1. rightward port from space 7 to 9
2. rightward port from space 4 to 6
3. rightward port from space 3 to 5

And we will find a contradiction in the α-value required of a starting army that achieves position S through the above moves.

We will first note the types of moves in PaSS for quick reference. Here are the types of moves that can achieve one counter on some arbitrary space i, the first two of which increase from zero to one counters, and the others decrease from two to one counters.

- Rightwards port from space $i-2$
- Leftwards port from space $i+2$
- Rightwards sweeps from spaces $i-2$, $i-1$, and i
- Leftwards sweeps from spaces i, $i+1$, and $i+2$

And here are the types of moves that can achieve two counters on space i, the last two of which increase from one to two counters.

- Rightwards sweep from space $i-3$
- Leftwards sweep from space $i+3$
- Rightwards port from space $i-2$
- Leftwards port from space $i+2$

Since $S(9) = 1$, a rightwards port from space 7 must have been made at some point to have arrived at position S. We will denote this position R.

$$\begin{array}{l} R - ? \mid ?\ ?\ 0\ 0\ 0\ 0\ 2\ 0\ 0 \\ S - ? \mid ?\ ?\ 0\ 0\ 0\ 0\ 0\ 0\ 1 \end{array}$$

Now, looking for the previous position, which we will call Q, all leftward port and sweep moves result in an unreachable position by Proposition 3. First, suppose that a rightwards port is used, which implies $Q(7) = 1$, $Q(5) = 2$, and no additional counters above space two. Moving backwards from there, we can ask which move resulted in $Q(7) = 1$. Again, any leftwards moves will contradict Proposition 3. Similarly, any rightwards sweeps that reduce that result in $Q(7) = 1$ result in unattainable positions. That leaves a rightwards port from space five again, which results in a previous board with too high of an α-value. Hence, the only move that could have resulted in the two counters on space seven is a rightwards sweep targeting cell 7. Hence, $Q(6) = Q(5) = Q(4) = 1$ and $Q(i) = 0$ for $i = 3$ and $i > 6$, the positions are shown below.

$$\begin{array}{l} Q - ? \mid ?\ ?\ 0\ 1\ 1\ 1\ 0\ 0\ 0 \\ R - ? \mid ?\ ?\ 0\ 0\ 0\ 0\ 2\ 0\ 0 \\ S - ? \mid ?\ ?\ 0\ 0\ 0\ 0\ 0\ 0\ 1 \end{array}$$

Now, we wish to show that we could only achieve Q such that $Q(6) = Q(5) = 1$ through rightwards ports. Looking at the moves that achieve one counter on space six, all leftward moves imply increases to the α-values of the previous position of at least 4.946, resulting in unreachable positions by Proposition 2. Rightwards sweeps targeting spaces 8 and 9 imply board positions that contradict Proposition 3. Lastly, a rightwards sweep targeting space seven implies a previous position P with $P(4) \geq 1$, $P(5) \geq 1$, and $P(6) = 2$. However $\alpha(P) \geq \alpha^4 + \alpha^5 + 2\alpha^6 > \mathcal{M}$, which is unreachable by Proposition 2. Hence, we must have used a rightwards port to get $Q(6) = 1$.

Now we will look at the moves that achieve 1 counter on space 5. Again, all leftwards moves imply increases to the α-values of the previous position of at least 4.009, resulting in unreachable positions by Proposition 3. A rightwards sweep targeting space eight implies a position with two counters on space eight which is unreachable by Proposition 3. A rightwards sweep targeting space seven implies a previous position P with $P(7) = 2$ and $P(5) = 1$ which yields $\alpha(P) \geq \mathcal{M}$, hence unreachable. In order to have used a rightwards sweep targeting space six, we would need a position with two counters on space six first which can only be achieved through leftwards moves ($Q(6) = 1$ and $Q(i) = 0$ for $i > 6$). However, the minimum increase in α-value of a leftwards move that increases the number of counters on space six is about 4.736 (a leftwards port), which implies an unreachable position. Hence, we must have also used a rightwards port to get the one counter on space five.

From the arguments above, in order to achieve a distance of nine from a valid starting army, the sequence of moves must include *a rightwards port to space 5*, *a rightwards port to space 6*, and *a rightwards port to space 9*. Let A denote the starting army, we can calculate an upper bound on $\alpha(A)$ by looking at the changes in α-value by the necessary moves above.

$$\begin{aligned}\alpha(A) &\leq \mathcal{M} + [\alpha\text{-effects of ports to 5, 6, and 9}] \\ &= \mathcal{M} + (\alpha^5 + \alpha^6 + \alpha^9)(1 - 2\alpha^{-2}) \\ &\approx 6.472\end{aligned}$$

However, this value is smaller than the value of any board with at least one counter on space nine ($\alpha^9 \approx 6.623$). Hence, no valid initial army can advance a distance of nine through legal PaSS moves.

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