Fourier Transform

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Fourier Transform is an important engineering tools that convert between a time-based function into a function of frequency. It is thought as one of the most important mathematics discoveries that has so many useful applications in multiple fields, including image processing, signal processing, electrical circuit design, ... In this paper, we will go through the motivation and the details of Fourier Transformation, and a brief introduction of the Discrete version of the Fourier Transform and its application.

1 Fourier Series

Taylor series and Laurent series are both great ways to represent functions. However, as we have seen in class, they both have their own limitations, where Taylor series only work well with holomorphic function, and Laurent series can work with function with finite isolated singularities.

In this section, we will introduce another representation of function, the Fourier Series. The Fourier Series are tools to represent periodic function as sums of sine and cosine waves.

Euler's formula gives us the equation for complex sine and cosine as follow:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2i}$$
$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

which can be used to deduce the formula for Fourier Series of function $f(x)$ for $-T < x < T$ is:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{in\frac{2\pi}{T}x}$$

where c_n is called the complex coefficients of the Fourier Series, which has the formula:

$$c_n = \int_T f(x)e^{-in\frac{2\pi}{T}x}dx$$

2 Continuous Fourier Transform

2.1 Fourier Tranformation

Fourier Transform is an extension of Fourier Series from representing only periodic function to turning any given time-based function to a function of frequency. Fourier Transform is a special case of Fourier Series, where $T \longrightarrow \infty$.

We start with the formula for the complex coefficient of the Fourier series:

$$c_n = \frac{1}{T} \int_T f(x) e^{-in\frac{2\pi}{T}x} dx$$
$$T \cdot c_n = \int_T f(x) e^{-in\frac{2\pi}{T}x} dx$$

Since $T \longrightarrow \infty$, the angular frequency $\frac{2\pi}{T}$ become very small, and the value $\frac{2n\pi}{T}$ becomes a continuous value that can take on any value ω. Substitute into the equation above, we have the analysis equation for the Fourier Transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

We can derive the inverse Fourier Transform using the same technique. First, we start with the synthesis formula for Fourier Series:

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n \cdot e^{in\frac{2\pi}{T}x} \\ &= T \sum_{n=-\infty}^{\infty} c_n \cdot e^{in\frac{2\pi}{T}x} \cdot \frac{1}{T} \end{aligned}$$

Since the angular frequency $\omega_0 = \frac{2\pi}{T}$, we have: $T = \frac{\omega_0}{2\pi}$. Substitute that and $F(\omega) = T \cdot c_n$ into the equation, we have:

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} F(\omega) e^{ix\omega} \frac{d\omega}{2\pi} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{ix\omega} d\omega \end{aligned}$$

which is the synthesis equation of the Fourier Transform, or the Inverse Fourier Transform.

2.2 Linearity of the Fourier Transformation

In this section, we provide the proof that Fourier Transformation is a linear transformation.

Theorem 1. *Show that the continuous Fourier Tranform is a linear transformation.*

Proof. Let $f(x) = \alpha \cdot f_1(x) + \beta \cdot f_2(x)$, apply the Fourier transformation to $f(x)$, we have:

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} (\alpha \cdot f_1(x) + \beta \cdot f_2(x)) e^{-i\omega x} dx \\ &= \int_{-\infty}^{\infty} (\alpha \cdot f_1(x)) e^{-i\omega x} + (\beta \cdot f_2(x)) e^{-i\omega x} dx \\ &= \int_{-\infty}^{\infty} (\alpha \cdot f_1(x)) e^{-i\omega x} + \int_{-\infty}^{\infty} (\beta \cdot f_2(x)) e^{-i\omega x} dx \\ &= \alpha \cdot \int_{-\infty}^{\infty} (f_1(x)) e^{-i\omega x} dx + \beta \cdot \int_{-\infty}^{\infty} (f_2(x)) e^{-i\omega x} dx \\ &= \alpha \cdot F_1(\omega) + \beta \cdot F_2(\omega) \end{aligned}$$

□

which completes out proof for the linearity of Fourier Transformation.

3 Discrete Fourier Transform (DFT)

In real life, we hardly have to deal with continuous value or infinity. Instead, we are faced with different discrete stream of time-based input, such as the radio signal, the pixels of each images, or the pressure of the sound waves. This prompts us to find a discrete transform to perform Fourier analysis. In this section, we will briefly introduce the definition of Discrete Fourier Transform and its practical applications.

Definition 1. *The Discrete Fourier Transform inputs a sequence of finite complex number $\{x_n\} = \{x_1, x_2, \cdots x_N\}$ and turn it into another sequence of complex number $\{X_k\} = \{X_1, X_2, \cdots X_N\}$ using the following formula:*

$$X_k = \sum_{i=0}^{N-1} x_i \cdot e^{\frac{nik2\pi}{N}}$$

The Discrete Fourier Transform is also an invertible linear transformation, and the Inverse Discrete Transform is given by:

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \cdot e^{i \frac{2\pi}{N} kn}$$

The Discrete Fourier Transform is studied intensively over the years, and many of its significant applications have been discovered by scientists and researchers. Some of its most prominent practical applications includes signal spectral analysis, data compression, fast polynomial/integer multiplication and its usefulness in solving partial differential equations.